$$g_2^{(m-1)} = \frac{1}{z^*} \sum_{q=1}^{m-1} [y_1^{(q)} y_3^{(m-q)} - \varkappa x_3^{(q)} (x_1^{(m-q)} + a_2 x_2^{(m-q)})]$$

$$g_3^{(m-1)} = \frac{1}{z^*} \sum_{q=1}^{m-1} [-y_1^{(q)} y_2^{(m-q)} + \varkappa x_3^{(q)} (x_2^{(m-q)} + a x_1^{(m-q)})]$$

Since series (13) converge absolutely for all $\varphi \geqslant \varphi_0$ and $b_1 < b^*$, it follows that when $\varphi \to \infty$, $x_s \to 0$, $y_s \to 0$ $(s = 1, 2, 3)$. This means that we can always choose the initial position and initial angular velocity of the Hess-Appel'rot gyroscope in such a manner, that its motion will tend, as time increases without limit, asymptotically to rotation about a horizontal axis. Such motions are called asymptotically pendulum-like motions.

We note that the class of asymptotically pendulum-like motions of the Hess-Appel'rot gyroscope described by relations (13) does not include the Hess solution as a special case. Indeed, the latter solution for the system of differential equations (6) is characterised by the invariant relation $x_1 = 0$. By virtue of the first equation of this system we find that if the relation $x_1 = 0$ holds at the initial instant, it holds at any other instant. For the class of asymptotically pendulum-like motions of the Hess-Appel'rot gyroscope and constant b_1 is not zero, and therefore $x_1 \neq 0$.

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INTERACTION OF THIRD-ORDER RESONANCES IN PROBLEMS OF THE STABILITY OF HAMILTONIAN SYSTEMS*

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The problem of the stability of the equilibrium state of a neutral Hamiltonian systems (all eigenvalues of the linearization matrices are purely imaginary) is considered. A stability criterion is obtained for systems with several third-order resonances.

1. Formulation of the problem. We shall study the stability of the equilibrium state of an autonomous Hamiltonian system of equations

$$x_\alpha{}^{\cdot} = \frac{\partial H(x, y)}{\partial y_\alpha}; \quad y_\alpha{}^{\cdot} = -\frac{\partial H(x, y)}{\partial x_\alpha}, \quad \alpha = 1, \ldots, N \tag{1.1}$$

$$H(x, y) = H_2(x, y) + H_3(x, y) + \ldots$$

$$x = (x_1, \ldots, x_N); \quad y = (y_1, \ldots, y_N)$$

Here $H_k(x, y)$ denotes the homogeneous k-th degree polynomials, Γ is the linearization matrix of the system, (1.1), $\lambda(\Gamma)$ are the eigenvalues and $\mathrm{Re}\,\lambda(\Gamma) = 0$.

We shall use the following definitions.

The system

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$$x^{\cdot} = F(x),\ F(0) = 0,\ \dim F = \dim x = 2N$$
$$\lambda(\Gamma) = \pm i\omega_j,\ j = 1, \ldots, N$$

has a resonance if an integral vector $k = (k_1, \ldots, k_N)$ exists such that $(k, \omega) = \Sigma k_j \omega_j = 0$. The number $|k| = \Sigma |k_j|$ will be called the order of the resonance and the vector k the resonance vector. If k_j are relatively simple, then $|k|$ has a minimum value.

Two resonances will be called independent if their resonance vectors are linearly independent. Two resonances are tied to m frequencies if their resonance relations contain exactly m general frequencies with non-zero components of the resonance vectors.

Below we study the stability of the equilibrium state of system (1.1) when it contains exactly two third-order resonances; we assume that there are no multiple roots $\lambda_\alpha \neq \lambda_j$ if $\alpha \neq j$. No other constraints are imposed on the system. The co-dimensionality of the systems in question out of all Hamiltonian systems with $\mathrm{Re}\,\lambda = 0$, is equal to two.

Let us formulate the results used in the present paper.

If system (1.1) has exactly one third-order resonance, then the following equivalent assertions hold /1, 2/.

Theorem 1a. The equilibrium state of the model system is unstable if no component of its resonance vector changes its sign; otherwise the system has a positive quadratic integral which guarantees the stability.

Here and henceforth by the model system we mean system (1.1) normalised to the third-order inclusive, with the higher-order terms neglected.

Theorem 1b. The equilibrium state of the model system is unstable if and only if its solutions include an increasing solution in the form of an invariant ray (for the definition of the incariant ray see /2/).

The following theorems hold for the complete initial system in the cases in question.

Theorem S. The stability of the model system implies the stability of the initial system up to the order k_0 ($k_0 > 3$ is the order of the lowest resonance; the resonances are assumed to be mutually independent).

Theorem NS. The instability of the model system implies the Lyapunov instability of the initial system.

The basic result of the present paper can be formulated as the following theorem.

Theorem 1. The model system is unstable if and only if the components of at least one resonance vector do not change their signs; otherwise the stability is guaranteed by the presence of a positive quadratic integral.

Notes. 1°. It is only the instability in the systems in question that is of course character and does not depend on the presence of higher-order resonances in the system.

2°. The presence of an increasing solution in the form of an invariant ray is not a necessary condition for the instability of the systems in question.

The proof of Theorem 1 is given in Sect.2. Theorem *NS* is proved in Sect.3, and the proof of theorem *S* is standard /3/ and therefore not given here.

Note that we can consider, without loss of generality, systems with the minimum possible number of degrees of freedom so that every frequency enters at least one of the resonance relations with non-zero component of the resonance vector.

2. Investigation of the model systems.

We use the non-degenerate canonical polynomial transformation to reduce the Hamiltonian (1.1) to its normal form up to and including the third-order terms, and neglect the higher-order terms. The resulting Hamiltonian describes the model system.

Proof of Theorem 1 for the case of unlinked resonances. In this case the Hamiltonian of the model system consists of the sum of two independent Hamiltonians, and the system splits into two independent subsystems each containing a single third-order resonance. This implies the validity of Theorem 1.

Proof of Theorem 1 for the case of the resonance linked to a single frequency. In this case the theorem holds for the general systems of differential equations /4/. However, in the case of general systems the two-frequency 1:2 resonance always leads (without additional degeneration) to instability, although this is not true for a Hamiltonian system. For a complete proof of Theorem 1 we must sort out two cases, specific for Hamiltonian systems only

1°. $$H_M{}^1 = -\rho_1 + 2\rho_2 + 4\rho_3 + 2A\sqrt{\rho_1{}^2\rho_2}\cos(\varphi_2 + 2\varphi_1) + 2B\sqrt{\rho_2{}^2\rho_3}\cos(\varphi_3 - 2\varphi_2)$$

2°. $$H_M{}^2 = \Sigma\omega_j\rho_j + 2A\sqrt{\rho_1{}^2\rho_2}\cos(\varphi_2 - 2\varphi_1) + 2B\sqrt{\rho_1\rho_3\rho_4}\cos(\varphi_1 + \varphi_3 + \varphi_4)$$

Here ρ_α and φ_α are canonical polar coordinates.

We note that the linking takes place in both cases with respect to the low frequency of the non-essential resonance.

The instability of a system with the Hamiltonian H_M^1 follows from the existence of an increasing solution in the form of an invariant ray

$$\psi_1 = \varphi_2 + 2\varphi_1 = -\frac{\pi|A|}{2A}, \quad \psi_2 = \varphi_3 - 2\varphi_2 = -\frac{\pi|B|}{2B}, \quad \rho_3 = b(t)$$
$$\rho_1 = \frac{4(2A^2+B^2)}{A^2} b(t), \quad \rho_2 = \frac{4A^2}{B^2} b(t), \quad b^{\cdot} = \frac{8A^2}{|B|} b^{3/2}$$

The instability of system with the Hamiltonian H_M^2 is proved in the same manner.

In the case of general systems (with additional degeneration) the situation is much more complicated, since generally speaking an increasing solution in the form of an invariant ray may not materialise.

Proof of Theorem 1 for the case when the resonances are linked with respect to two frequencies. Two-frequency and three-frequency resonances. Let $|\omega_1| < |\omega_2| < |\omega_3|$. We can assume without loss of generality that $|\omega_1| = 1$ when $|\omega_2| = 2$, $|\omega_3| = 3$. When H is indefinite, three cases are possible apart from sign: $H_2 = -\rho_1 + 2\rho_2 + 3\rho_3$, 2) $H_2 = \rho_1 - 2\rho_2 + 3\rho_3$, 3) $H_2 = \rho_1 + 2\rho_2 - 3\rho_3$

It can be confirmed that in cases 1) and 3) the instability follows from the existence of an increasing solution in the form of an invariant ray. Let us carry out the proof for the more complicated case 2). We have

$$H = \rho_1 - 2\rho_2 + 3\rho_3 + 2F_1(\rho, \psi_1) + 2F_2(\rho, \psi_2) \tag{2.1}$$
$$\psi_1 = \varphi_2 + 2\varphi_1, \ \psi_2 = \varphi_3 + \varphi_2 - \varphi_1, \ F_1 = \sqrt{\rho_1^2\rho_2}\cos\psi_1$$
$$F_2 = \sqrt{\rho_1\rho_2\rho_3}\cos\psi_2, \ G_1 = \sqrt{\rho_1^2\rho_2}\sin\psi_1, \ G_2 = \sqrt{\rho_1\rho_2\rho_3}\sin\psi_2$$
$$\rho_1^{\cdot} = 2(-G_1 + G_2), \ \rho_2^{\cdot} = -2(G_1 + G_2), \ \rho_3^{\cdot} = -2G_2$$
$$\psi_1^{\cdot} = (-F_1(\rho_1 + 4\rho_2) + 2F_2(2\rho_2 - \rho_1))/(\rho_1\rho_2)$$
$$\psi_2^{\cdot} = F_1(2\rho_2 - \rho_1)/(\rho_1\rho_2) + 2F_2(1/\rho_1 - 1/\rho_2 - 1/\rho_3)$$

and $(I = \rho_1 - 2\rho_2 + 3\rho_3$, $K = H - I$ are integrals of the system (2.1).

We will seek the solution of the system in the form of a ray on the invariant surface

$$I = K = 0, \quad \psi_1 = -\frac{\pi|A|}{2A}, \quad \psi_2 = -\frac{\pi|B|}{B}, \quad \rho_2 = \frac{1}{2}(\rho_1 + 3\rho_3)$$

and for the remaining variables we obtain

$$\rho_1^{\cdot} = (2\sqrt{2}|A|\rho_1 - \sqrt{2}|B|\sqrt{\rho_1\rho_3})\sqrt{\rho_1 + 3\rho_3} \tag{2.2}$$
$$\rho_3^{\cdot} = \sqrt{2}|B|\sqrt{\rho_1\rho_3(\rho_1 + 3\rho_3)}$$

We seek the increasing solution in the form $\rho_3(t) = k\rho_1(t)$, $k > 0$. Such a solution exists, provided that a positive solution of the equation $2|A| = |B|(k + 1/k)$ exists. Thus when $|A| \geqslant |B|$, the increasing solution has the form

$$\rho_1^{\cdot} = \sqrt{2}|B|\frac{\sqrt{1+3k}}{\sqrt{k}}\rho_1^{3/2}$$

There is no increasing solution in the form of a ray when $|A| < |B|$. We shall show that in this case system (3.2), and hence (3.1), are both unstable.

Let us put

$$F = \rho_1 + \frac{2A^2 + B^2}{B^2}\rho_3 - \frac{|A|}{|B|}\sqrt{\rho_1\rho_3}$$

If $dF/dt > 0$, then F is a Chetaev function of system (2.2) and the instability is proved.

We note that the Chetaev function constructed guarantees the instability of system (2.1) irrespective of the sign of $|A| - |B|$.

Two three-frequency resonances. Let

$$k = (k_1, k_2, k_3, 0), \ l = (0, l_2, l_3, l_4), \ |k_j| = |l_{j+1}| = 1$$

be the resonance vectors of the system. If a change of sign occurs amongst the non-zero components of both vectors, then a positive vector $p = (p_1, p_2, p_3, p_4)$, $p_j > 0$ exists such that $(\rho, k) = (p, l) = 0$ and the positive quadratic integral $I = (p, l)$ /5/ in this case guarantees the stability of the model system.

Let us now assume that there is no change of sign amongst the components of one of the resonance vectors (e.g. k). Then $l_2 l_3 < 0$, otherwise the system would have multiple roots ($|\omega_1| = |\omega_4|$). Thus l can have the form $l = (0, 1, -1, 1)$ or $l = (0, 1, -1, -1)$, apart from the sign.

Let us consider the first case (the second case can be tackled in exactly the same manner). We have

$$H = \Sigma\omega_\alpha\rho_\alpha + 2A\sqrt{\rho_1\rho_2\rho_3}\cos\psi_1 + 2B\sqrt{\rho_2\rho_3\rho_4}\cos\psi_2$$
$$\omega_1 + \omega_2 + \omega_3 = 0, \ \varphi_1 + \varphi_2 + \varphi_3 = \psi_1, \ \omega_2 - \omega_3 + \omega_4 = 0,$$
$$\varphi_2 - \varphi_3 + \varphi_4 = \psi_2$$

The integrals of the system are

$$I_1 = 2\rho_1 - \rho_2 - \rho_3,\ I_2 = 2\rho_4 + \rho_3 - \rho_2,\ K = H - \Sigma\omega_\alpha\rho_\alpha$$

Consider the invariant manifold

$$\Omega:\ I_1 = I_2 = K = 0$$
$$\psi_1 = -\frac{\pi|A|}{2A},\quad \psi_2 = -\frac{\pi|B|}{2B},\quad \rho_1 = \frac{1}{2}(\rho_4+\rho_3),\quad \rho_2 = \rho_3 + 2\rho_4$$

For the remaining variables we obtain on Ω

$$\rho_3^{\cdot} = (\sqrt{2}\,|A|\,\sqrt{(\rho_3+\rho_4)\,\rho_3} - 2\,|B|\,\sqrt{\rho_3\rho_4})\,\sqrt{\rho_3+2\rho_4}$$
$$\rho_4^{\cdot} = 2\,|B|\,\sqrt{\rho_3\rho_4\,(\rho_3+2\rho_4)}$$

If $|A| > \sqrt{2}\,|B|$, then the system has an increasing solution in the form of a ray

$$\rho_3 = k\rho_4(t),\quad k = \frac{A^2-2B^2}{B^2},\quad \rho_4^{\cdot} = 2\,|B|\,\sqrt{k(k+2)}\,\rho_4^{3/2}$$

and when $|A| \leqslant \sqrt{2}\,|B|$, there is no such solution.

Let us consider the function

$$F = (\sqrt{\rho_4} - \sqrt{\rho_3})/\,|B|$$

We have

$$dF/dt = (\sqrt{\rho_3} + \sqrt{\rho_4} - k^2\sqrt{\rho_3+\rho_4})\,\sqrt{\rho_3+2\rho_4} \geqslant$$
$$(1-k^2)\,(\sqrt{\rho_3}+\sqrt{\rho_4})\,\sqrt{\rho_3+2\rho_4};\ k_2 = |A|/(\sqrt{2}\,|B|) < 1$$

therefore F is a Chetaev function.

We have here used the inequality $a + b - k^2\sqrt{a^2+b^2} \geqslant \frac{1}{2}(1-k^2)(a+b)$.

3. On the non-essential nature of the higher-order terms. The results of Sect.2 imply that the instability in the model systems in question appears as a result of: a) the presence of the increasing solution in the form of an invariant ray, or b) the presence of the increasing solutions whose form is unknown and whose existence follows form the existence of the Chetaev function.

The proof of the non-essential nature of the higher-order terms was studied in detail for the case a) in e.g. /3/, and is not given here.

In the case b) the situation is less clear. In proving that the higher-order terms are not essential we can use the same Chetaev function as in the model systems. Since cases 1) and 3) in Sect.2 are the same, we shall give the proof only for one of them, namely for the case where the two-frequency and three-frequency resonances are linked to two frequencies.

The system in question is identical in its principal terms with system (2.1). Let us rewrite it, expanding the right-hand sides in a Taylor series near the point

$$\psi_1^\circ = -\frac{\pi|A|}{2A},\quad \psi_2^\circ = -\frac{\pi|B|}{2B}$$

We obtain

$$\rho_1^{\cdot} = 4\,|A_1|\,\sqrt{\rho_1^2\rho_2} - 2\sqrt{\rho_1\rho_2\rho_3} + P(\rho,\psi)$$
$$\rho_2^{\cdot} = 2\,|A_1|\,\sqrt{\rho_1^2\rho_2} + 2\sqrt{\rho_1\rho_1\rho_3} + \rho(\rho,\psi)$$
$$\rho_3^{\cdot} = 2\sqrt{\rho_1\rho_2\rho_3} + P(\rho,\psi)$$
$$\psi_1^{\cdot} = -2\,|A_1|\,\sqrt{\rho_2}\,\psi_1 + O(\sqrt{\rho}\,|\psi|^2) + O(\rho)$$
$$\psi_2^{\cdot} = -\frac{\sqrt{\rho_2}}{\sqrt{\rho_1\rho_3}}(\rho_1+\rho_3)\,\psi_2 + O(\sqrt{\rho}\,|\psi|^2) + O(\rho)$$
$$(P(\rho,\psi) = O(\rho^{3/2}\,|\psi|) + O(\rho^2),\ dt_1 = dt/\,|B|,\ A_1 = A/\,|B|)$$

The variable changes in this section are not canonical.

Let us introduce the following spherical coordinates for the variables ρ_j: $\rho_2 = R\sin^2\theta_1$, $\rho_1 = 2R\cos^2\theta_1\sin^2\theta_2$, $\rho_3 = {}^2/_3R\cos^2\theta_1\cos^2\theta_2$. This yields

$$dR/d\tau = \Pi(R,\ \theta_2) + O(R(|\theta_1| + |\psi|)) + O(R^{3/2}) \tag{3.1}$$
$$d\theta_2/d\tau = G_1(\theta_2) + O(|\theta| + |\psi|) + o(\sqrt{R})$$
$$d\theta_1/d\tau = -(2\sqrt{2}/3)(1+\sqrt{2}\,|A_1|)\sin\theta_2\cos\theta_2\theta_1 + O(|\theta_1|^2 + |\psi|) + O(\sqrt{R})$$
$$d\psi_1/d\tau = -\sqrt{2}\,|A_1|\,\psi_1 + O(|\theta_1|^2 + |\psi|^2) + O(\sqrt{R})$$
$$\frac{d\psi_2}{d\tau} = -\sqrt{\frac{3}{2}}\,\frac{(\cos^2\theta_2 + 3\sin^2\theta_2)}{\sin\theta_2\cos\theta_2} + O(|\theta_1|^2 + |\psi|^2) + O(\sqrt{R})$$
$$\Pi = 4\sqrt{2}\,R\sin\theta_2\,(|A_1|\sin\theta_2 + \cos O_2/\sqrt{3})$$
$$G_1 = (3/\sqrt{2})\,((2\,|A_1|/\sqrt{3})\sin\theta_2\cos\theta_2 - {}^1/_3\cos^2\theta_2)$$

Here we have made the change of variable $d\tau = \sqrt{R}\,dt$, which is valid provided that

$$\int\sqrt{R(t)}\,dt$$

diverges. In the present situation this is true, and it follows for example, from the following lemma.

Lemma. The integral $\int_0^\infty \sqrt{R(t)}\,dt$ diverges on any solution $R(t), R(0) > 0$ of the equation

$$dR/dt = R^{3/2}\Pi(\theta(t)),\ |\Pi| < p^2$$

We recall that when $|A_1| < 1$, then the model system has no increasing ray and this implies that there is no stationary point in the angular system.

Let the manifold

$$\Omega = \{R, \theta, \psi: |\theta_1| + |\psi_1| + |\psi_2| < \varepsilon,\ R < R^\circ,\ 0 \leqslant \theta_2 \leqslant \pi/2\}$$

where Ω is invariant with respect to system (3.1); the trajectories only enter this region.

Let us consider the function

$$F = R\left[\sin^2\theta_2 + \frac{1+3A_1^2}{3}\cos^2\theta_2 - \frac{A_1}{\sqrt{3}}\sin\theta_2\cos\theta_2\right]$$

Calculating $dF/d\tau$ on Ω we obtain

$$dF/d\tau = |A_1| R[1 - \beta\varepsilon + O(\sqrt{R})] > 0$$

This implies that Ω, F is a Chetaev pair of the first kind and the theorem is proved.

4. A theorem on the stability of the systems close to the critical systems in question. Let us consider a system with the Hamiltonian

$$H(x, y) = H_0(x, y) + \alpha H_1(x, y),\ H(0, 0) = 0 \tag{4.1}$$

Here $H(x, y)$ is a smooth function and the expansion in a Taylor series begins with the second-order terms. Let $H_0(x, y)$ be the Hamiltonian of one of the critical cases discussed above.

Theorem. Let a system corresponding to the Hamiltonian H_0 be unstable when $\alpha = 0$. Then for every $\alpha\ (|\alpha| < \varepsilon)$ a solution $u_\alpha(t)$ of system (4.1) exists for which $|u_\alpha(0)| < c|\alpha|^\varkappa$, $\sup_{t>0} |u_\alpha(t)| > b$ (b is independent of α), $\varkappa = 1$.

The proof of the existence of $\varkappa = 1$ is standard /3/ and will not be given here. The proof of the fact that index cannot be improved upon is as follows. We choose a perturbation $\alpha H_1(x, y)$ so that the resonance relations are satisfied apart from ε: $(k\omega) = (l\omega) = \varepsilon$. Then a positive integral exists (of a resonance-free problem) acting in the complete ε-neighbourhood of the zero. Consequently a "patch" $\Omega\ (d\Omega \sim \varepsilon)$ exists, and this guarantees that the index $\varkappa = 1$ cannot be improved.

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